ARTICLES

Anomalous diffusion resulting from strongly asymmetric random walks

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We present a model of one-dimensional asymmetric random walks. Random walkers alternate between flights (steps of constant velocity) and sticking (pauses). The sticking time probability distribution function (PDF) decays as $P(t)\sim t^{-\nu}$. Previous work considered the case of a flight PDF decaying as $P(t)\sim t^{-\mu}$ [Weeks *et al.*, Physica D **97**, 291 (1996)]; leftward and rightward flights occurred with differing probabilities and velocities. In addition to these asymmetries, the present strongly asymmetric model uses distinct flight PDFs for leftward and rightward flights: $P_L(t)\sim t^{-\mu}$ and $P_R(t)\sim t^{-\eta}$, with $\mu\neq\eta$. We calculate the dependence of the variance exponent γ ($\sigma^2\sim t^\gamma$) on the PDF exponents ν, μ, and η. We find that γ is determined by the two smaller of the three PDF exponents, and in some cases by only the smallest. A PDF with decay exponent less than 3 has a divergent second moment, and thus is a Lévy distribution. When the smallest decay exponent is between 3/2 and 3, the motion is superdiffusive ($1<\gamma<2$). When the smallest exponent is between 1 and 3/2, the motion can be subdiffusive ($\gamma<1$); this is in contrast with the case with $\mu=\eta$. [S1063-651X(98)01205-7]

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I. INTRODUCTION

Random walks are closely tied with diffusive processes. Einstein showed in 1905 [1] that normal diffusion of tracers in liquids is a result of the random walks of the tracers due to Brownian motion, and that the variance of the ensemble of tracers spreads as

$$\sigma^2(t)=\langle x^2(t)\rangle-\langle x(t)\rangle^2=2Dt, \tag{1}$$

where D is the diffusion constant [$x(t)$ represents one-dimensional motion]. If the tracer particles take steps of length l, with a probability distribution function (PDF) for step lengths $P(l)$, the central limit theorem shows that

$$D=\frac{\langle l^2\rangle-\langle l\rangle^2}{2T}, \tag{2}$$

where $\langle l^n\rangle$ are the moments of $P(l)$ and T is a characteristic time between each step [2].

The central limit theorem does not apply for random walks known as Lévy flights. For Lévy flights, the step length PDF decays as $P(l)\sim l^{-\mu}$ for large l, with $\mu<3$. In such cases, $\langle l^2\rangle=\infty$ and the diffusion constant is effectively infinite. Such broad PDFs with slow decay are known as Lévy distributions [2,3]. (Some authors use the term Lévy flights for random walks with instantaneous steps, and use Lévy walks for situations with steps of finite velocity; we will use Lévy flights for both cases.) For Lévy flights, the variance grows *superdiffusively* rather than diffusively: $\sigma^2(t)\sim t^\gamma$, with $1<\gamma<2$. This behavior has been seen in a variety of experiments, including tracers in fluid flow with jets and vortices [4–7], particles carried by capillary waves [8,9], tracers in the ocean [10], and mixing of polymerlike micelles [11,12].

In other situations, $\langle l^2\rangle$ may be finite but T, the time between steps, may be infinite, and thus $D=0$ [by Eq. (2)]. In this situation, $\sigma^2(t)\sim t^\gamma$ with $0<\gamma<1$, which is termed *subdiffusion*. Subdiffusion can arise when the tracer particles pause between steps, with a pause time (also known as a sticking time) PDF decaying as $P(t)\sim t^{-\nu}$ for large t. If $\nu<2$, the mean sticking time is infinite; thus the time per step T is infinite. In experiments this sticking behavior often results from particles being captured in vortices [4–7,13–15] or being locally trapped by some sort of potential well [16,17]. Subdiffusive behavior has been observed in arrays of vortices of alternating sign [13–15] and in observations of the photoconductivity of amorphous materials [16,17].

Lévy flights and broad sticking PDFs have been useful for the study of Hamiltonian systems and conservative maps. Coherent structures within chaotic phase space can lead to long sticking times [18–23] or long flights [3,22–26]. A recent paper found a Hamiltonian model that has two different types of flights with different PDFs, which is the case considered in this paper (strong asymmetry) [27]. Lévy flights also appear in an analysis of subrecoil laser cooling of atoms [28,29], where the mean time for atoms to leave an optical trap is infinite.

In general, when the central limit theorem no longer applies, it is still possible to determine analytically the value of the variance exponent γ. Quantitative connections between the behavior of the distribution functions and the exponent γ have been made for symmetric random walks [7,25,30–34]. In addition, several authors have considered asymmetric ran-

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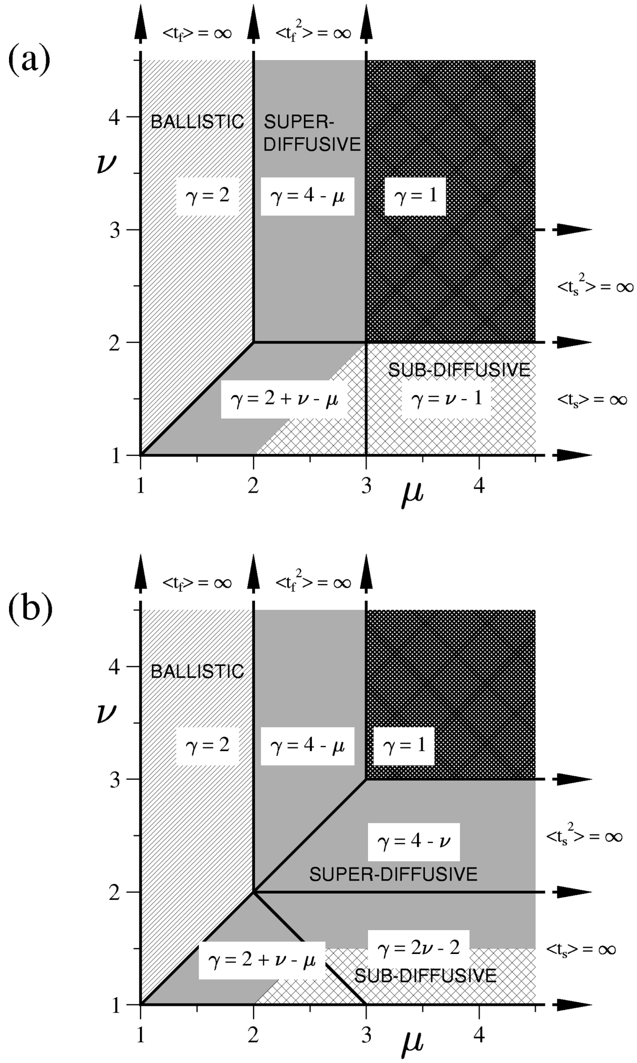

FIG. 1. Previously derived phase diagrams for variance exponent γ $[\sigma^2(t) \sim t^\gamma]$ of (a) symmetric and (b) asymmetric random walks. The horizontal axis is μ, the exponent for the flight PDF: $P_F(t) \sim t^{-\mu}$ for large t; the flight PDF is identical for leftward and rightward flights, in contrast to Fig. 2. The vertical axis is ν, the exponent for the sticking PDF: $P_S(t) \sim t^{-\nu}$ for large t. For each region bordered by the solid lines, the relationship between the variance exponent γ and PDF exponents μ and ν is shown. The shadings indicate areas where the behavior is subdiffusive ($\gamma < 1$), normally diffusive ($\gamma = 1$), superdiffusive ($1 < \gamma < 2$), and ballistic ($\gamma = 2$). From Ref. [7].

dom walks, where the probability to take leftward steps is different from the probability to take rightward steps [7,33]. This sort of asymmetry occurs in situations such as charged particles moving in an electric field, or particles carried by an asymmetric flow. The results from these authors for the symmetric and asymmetric cases are summarized in Ref. [7], and in Fig. 1.

Previous work considered cases where the probability to take leftward and rightward flights could be different [7,33]. Here we additionally consider the case where the decay exponents for the flight PDF are different for leftward and rightward flights, which we term ''strong asymmetry.'' We examine the case of flights with finite velocity. We consider only the case of a continuous time random walk, where distinct steps are taken with constant velocity separated by pauses. Other mechanisms [2,35–37] that may lead to anomalous diffusion $[\sigma^2(t) \sim t^\gamma, \gamma \neq 1]$ are reviewed in Ref. [2].

II. ASYMMETRIC RANDOM WALKS

In this section we use a model based on a one-dimensional random walk to predict the asymptotic scaling of the variance: as $t \to \infty$, $\sigma^2(t) \sim t^\gamma$. The goal is to find the dependence of γ on the parameters of the model, and to examine differences between symmetric, asymmetric, and the current case of strongly asymmetric random walks.

A. Model

We consider a random walker alternating between sticking (remaining at the same location for some length of time) and constant velocity flights. The flights consist of two types: flights of velocity v_l, with a distribution function $P_{Fl}(t)$, and flights of velocity v_r, with a distribution $P_{Fr}(t)$. If $v_l < 0$ and $v_r > 0$ then the flights are in the leftward and rightward directions, respectively; if v_l and v_r have the same sign, the flights are in the same direction. When ending a sticking event, the probability of a leftward flight is p_l, and the probability of a rightward flight is $p_r = 1 - p_l$. The random walker begins at the origin $x = 0$, and at time $t = 0$ begins a flight (with probability p_F^0) or a sticking event (with probability $p_S^0 = 1 - p_F^0$). The duration of sticking events is given by the PDF $P_S(t)$. For the moment, we make no assumptions about the forms of the flight or sticking PDFs.

Our goal is to find the PDF $X(x,t)$ of the random walker position for large times, following a procedure similar to that of Refs. [7,31]. From this PDF we can calculate the variance, $\sigma^2(t) = \langle x^2(t) \rangle - \langle x(t) \rangle^2$, and extract the scaling exponent γ. The moments of x are obtained from the Fourier transform of X:

$$(i^n) \frac{\partial^n \tilde{X}(k,t)}{\partial k^n} \Big|_{k=0} = \langle x^n \rangle. \tag{3}$$

We construct $\tilde{X}(k,t)$ from simpler PDFs related to the particle motion. We need $\xi(x,t)$, the probability that a flight event has a distance of x and a duration of t:

$$\xi(x,t) = p_r \delta(x - v_r t) P_{Fr}(t) + p_l \delta(x - v_l t) P_{Fl}(t). \tag{4}$$

The Dirac δ functions ensure that the flights are made with the correct constant velocity. Following the method of Ref. [7], we find the Fourier-Laplace transform (in space and time, respectively) of $X(r,t)$ to be

$$\tilde{X}(k,s) = \{s^{-1}[1 - \tilde{P}_S(s)]\} \left[\frac{p_S^0 + p_F^0 \tilde{\xi}(k,s)}{1 - \tilde{\xi}(k,s) \tilde{P}_S(s)} \right] + [p_r \tilde{\lambda}_r + p_l \tilde{\lambda}_l] \left[\frac{p_F^0 + p_S^0 \tilde{P}_S(s)}{1 - \tilde{\xi}(k,s) \tilde{P}_S(s)} \right] \tag{5}$$

where the function $\tilde{\lambda}$ has been introduced for convenience:

$$\widetilde{\lambda}_r(s)=p_r(s+ikv_r)^{-1}[1-\widetilde{P}_{Fr}(s+ikv_r)] \tag{6}$$

and similarly for $\widetilde{\lambda}_l(s)$. Noting that

$$\widetilde{\xi}(k,s)=p_r\widetilde{P}_{Fr}(s+ikv_r)+p_l\widetilde{P}_{Fl}(s+ikv_l), \tag{7}$$

we have $\widetilde{X}$ expressed completely in terms of the three elementary PDFs, $\widetilde{P}_S(s)$, $\widetilde{P}_{Fl}(s)$, and $\widetilde{P}_{Fr}(s)$.

Using Eq. (3), we obtain $\langle x\rangle$ and $\langle x^2\rangle$ by taking derivatives of $\widetilde{X}(k,s)$:

$$\langle x\rangle=\frac{(p_F^0+p_S^0\widetilde{P}_S)[\widetilde{Z}_r+\widetilde{Z}_l]}{s^2(1-\widetilde{\xi}\widetilde{P}_S)} \tag{8}$$

$$\langle x^2\rangle=[(1-\widetilde{\xi}\widetilde{P}_S)(\widetilde{Y}_r+\widetilde{Y}_l)-s\widetilde{P}_S(\widetilde{Z}_r+\widetilde{Z}_l)(p_rv_r\widetilde{P}'_{Fr}+p_lv_l\widetilde{P}'_{Fl})]\frac{2(p_F^0+p_S^0\widetilde{P}_S)}{s^3(1-\widetilde{\xi}\widetilde{P}_S)^2}, \tag{9}$$

where

$$\widetilde{Y}_r(s)=p_rv_r^2(1-\widetilde{P}_{Fr}+s\widetilde{P}'_{Fr}), \tag{10}$$

$$\widetilde{Z}_r(s)=p_rv_r(1-\widetilde{P}_{Fr}), \tag{11}$$

and similarly for $Y_l(s)$ and $Z_l(s)$. If $P_{Fl}(t)=P_{Fr}(t)$, these results reduce to Eqs. (27) and (28) of Ref. [7]. The results in Eqs. (8) and (9) are exact for any form of $P_{Fl,r}(t)$ and $P_S(t)$; no approximations have been made.

B. Results

The asymptotic behavior of $\langle x(t)\rangle$ and $\langle x^2(t)\rangle$ as $t\to\infty(s\to 0)$ can be obtained from an expansion of Eqs. (8) and (9) in powers of s. This depends on $\widetilde{P}_{Fl,r}(s)$ and $\widetilde{P}_S$, which in turn depend on the particular form of $P(t)$ for these functions. After expanding for small s, the leading terms can be inverse Laplace transformed to find the behavior for large t.

We choose flight and sticking PDFs to be of the form [7]

$$P(t)=\begin{cases}0, & t<t_\alpha\\ A_\alpha t^{-\alpha}, & t\geqslant t_\alpha,\end{cases} \tag{12}$$

where α is either η, μ, or ν for P_{Fl}, P_{Fr}, and P_S, respectively, and t_α is a cutoff at short times to allow the function to be normalizable; the normalization constant is $A_\alpha=(\alpha-1)t_\alpha^{\alpha-1}$. The cutoff times t_η, t_μ, and t_ν may be different. The scaling exponent γ of the variance only depends on the asymptotic behavior of the sticking and flight PDFs (the exponents η, μ, and ν), although some results that follow will depend slightly on the behavior at short times; this will be clarified later.

The Laplace transforms of the PDFs $P(t)$ [Eq. (12)] have the form

$$\widetilde{P}(s)=A_\alpha s^{\alpha-1}\Gamma(1-\alpha,st_\alpha). \tag{13}$$

Expanding the incomplete Γ function for small arguments yields

$$\widetilde{P}(s)=-\Gamma(2-\alpha)t_\alpha^{\alpha-1}s^{\alpha-1}+1-\langle t\rangle s+\frac{1}{2!}\langle t^2\rangle s^2-\frac{1}{3!}\langle t^3\rangle s^3+\cdots. \tag{14}$$

The expression in terms of the moments $\langle t\rangle$ of the PDFs is correct only for these particular PDF's.

TABLE I. Scaling of the mean position. $\langle x\rangle\sim Kt^\beta$. β is correct for any PDF with the same asymptotic scaling, while the values shown for K are correct only for the specific form of the PDFs [Eq. (12)]. Without loss of generality, we assume that μ is the smaller of the two flight decay exponents. ν is the decay exponent for the sticking PDF. In the expressions for K, $\langle l\rangle=p_lv_l\langle t_{Fl}\rangle+p_rv_r\langle t_{Fr}\rangle$ and $T=p_l\langle t_{Fl}\rangle+p_r\langle t_{Fr}\rangle+\langle t_S\rangle$. These results are similar to those given in Ref. [7].

Conditions		β	Coefficient K
$\mu>2$	$\nu>2$	1	$\langle l\rangle/T$
$1<\mu<2$	$\nu>\mu$	1	v_r
$\mu>2$	$1<\nu<2$	$\nu-1$	$\left(\frac{\langle l\rangle}{\Gamma(2-\nu)\Gamma(\nu)}\right)t_\nu^{1-\nu}$
$1<\mu<2$	$1<\nu<\mu$	$1+\nu-\mu$	$\left(\frac{v_r\Gamma(2-\mu)}{\Gamma(2-\nu)\Gamma(2-\mu+\nu)}\right)p_rt_\mu^{\mu-1}t_\nu^{1-\nu}$

We start by computing the behavior of the mean, $\langle x\rangle\sim Kt^\beta$, using Eq. (8). The results are presented in Table I. These results can be understood in relationship to the underlying PDFs. When all PDF exponents are larger than 2, the mean grows proportional to $\langle l\rangle/T$, that is, the mean step displacement divided by the time between steps. When a flight exponent is less than 2, the mean flight time is infinite. In this situation, for an ensemble of random walkers, for any time t the typical random walker is still undergoing its first flight, so the mean position for the ensemble of walkers grows as v_r, the velocity of those walkers (assuming $\mu<\eta$; otherwise the relevant velocity is v_l). When the sticking PDF has an infinite first moment ($\nu<2$), the mean position grows slower than linearly in time, with the growth dependent on the flight behavior; on average, random walkers are undergoing their first sticking event, and the growth of the mean is dependent on the rare walkers not sticking. Cases with $\beta\neq 1$ are termed *anomalous advection* [38].

Similarly, we expand Eq. (9) using Eq. (14) to find $\langle x^2\rangle$, and ultimately to find $\sigma^2(t)=\langle x^2\rangle-\langle x\rangle^2$. The results are shown in Fig. 2 and Table II. The results depend only on the smallest two PDF decay exponents, but are symmetric between flight and sticking behavior (except for the $\gamma=1$ coefficient). The variance growth exponent γ in particular is determined by the smallest exponent, and in some cases the second smallest exponent as well (see Table II). Note that the transitions from one phase to another that occur as the exponents of the PDFs are varied are sharply defined only in the infinite time limit.

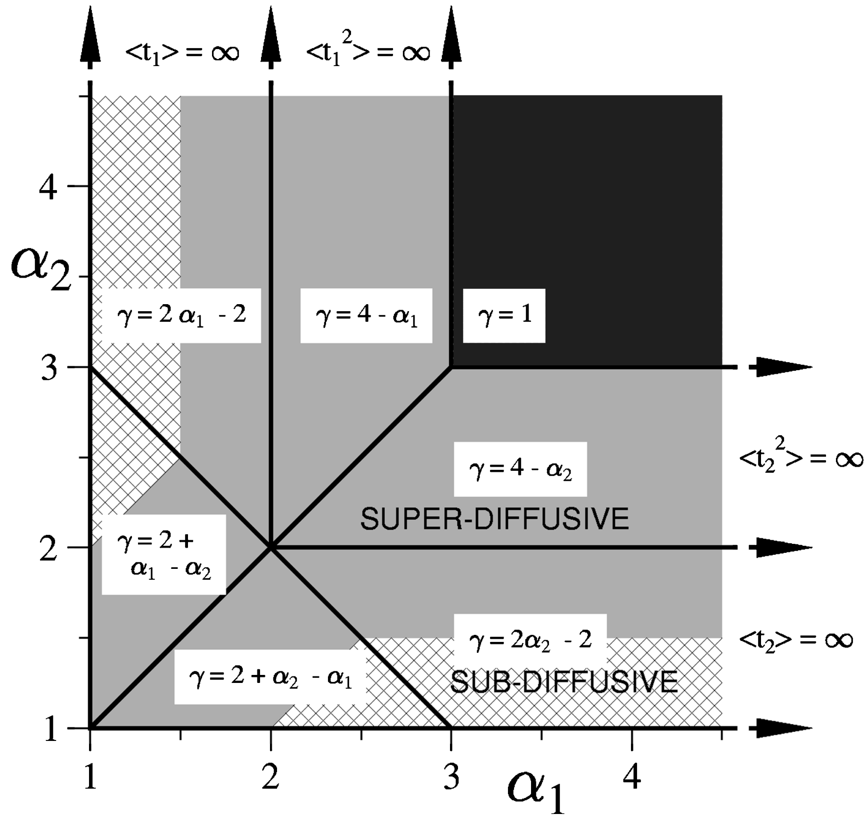

FIG. 2. Phase diagram for the variance exponent γ [$\sigma^2(t) \sim t^\gamma$] for the strongly asymmetric case; the flight PDFs have distinct decay exponents. α_1 and α_2 are the exponents controlling the asymptotic power-law decay of the sticking PDF and the two flight PDFs: $P(t) \sim t^{-\alpha}$ for $t \rightarrow \infty$. Of the three PDFs controlling the behavior (flights to the left, flights to the right, sticking), α_1 and α_2 are the lowest two exponents, although by symmetry of the results for this graph it does not matter which exponent is the smallest. For each region, bordered by the solid lines, the relationship between the variance exponent γ, α_1, and α_2 is shown. The shadings indicate areas where the behavior is normally diffusive ($\gamma=1$), subdiffusive ($\gamma<1$), and superdiffusive ($\gamma>1$). Compare with Fig. 1(b); the main difference is that the strongly asymmetric case shown in this figure does not have a ballistic region.

TABLE II. Anomalous diffusion results: $\sigma^2 \sim Ct^\gamma$, with the coefficients C and exponent γ given in the table. The α_i's represent the sorted decay exponents of the PDFs ($\alpha_1<\alpha_2<\alpha_3$). The variable p_i represents $(p_l, p_r, 1)$ if α_i corresponds with (η, μ, ν), respectively; similarly v_i represents $(v_l, v_r, 0)$. τ_i is defined in terms of the cutoff times [see Eq. (12)]: $\tau_i = (t_{Fl}, t_{Fr}, t_S)$ if α_i corresponds with (η, μ, ν), respectively. $T = \Sigma_{j=1}^{3} p_j \langle t_j \rangle$ and $L_i = \Sigma_{j=1}^{3} p_j (v_i - v_j)\langle t_j \rangle$. (Note that this definition for T is equivalent to the one given in Table I.) γ is correct for all PDFs with the same asymptotic scaling, while the values shown for C are correct only for the specific form of the PDFs [Eq. (12)].

Conditions		γ	Coefficient C
$\alpha_1>3$	$\alpha_2>3$	1	$-2\langle t_s\rangle^2\langle l\rangle+\sum_{i=1}^{3} p_i \frac{\langle t_i^2\rangle}{T^3} L_i^2$
$2<\alpha_1<3$	$\alpha_2>\alpha_1$	$4-\alpha_1$	$\frac{2L_1^2}{(4-\alpha_1)(3-\alpha_2)}\left(\frac{p_1\tau_1^{\alpha_1-1}}{T^3}\right)$
$\alpha_2>4-\alpha_1$	$1<\alpha_1<2$	$2\alpha_1-2$	$\left(\frac{2\Gamma^2(\alpha_1)-\Gamma(2\alpha_1-1)}{\Gamma^2(2-\alpha_1)\Gamma^2(\alpha_1)\Gamma(2\alpha_1-1)}\right)L_1^2(p_1\tau_1^{\alpha_1-1})^{-2}$
$\alpha_1<\alpha_2<4-\alpha_1$	$1<\alpha_1<2$	$2+\alpha_1-\alpha_2$	$\left(\frac{2\Gamma(3-\alpha_2)}{\Gamma(2-\alpha_1)\Gamma(3+\alpha_1-\alpha_2)}\right)\frac{p_2\tau_2^{\alpha_2-1}}{p_1\tau_1^{\alpha_1-1}}(v_1-v_2)^2$

As is the case for the mean $\langle x \rangle$ discussed above, the behavior can be understood through the PDFs. Let α_1 be the smallest of the PDF decay exponents. If $\alpha_1 > 3$ then all three PDFs have finite first and second moments, and the central limit theorem must apply. In this case the growth is normally diffusive, that is, $\gamma = 1$. If $2 < \alpha_1 < 3$, the second moment for that PDF is infinite; this is the situation where flights are Lévy flights. The mean position scales as $\langle x \rangle \sim t$, and the growth in the variance results from spreading about this mean position. As expected for situations with Lévy flights, the growth is superdiffusive. If the smallest exponent is the sticking exponent (that is, if $\alpha_1 = \nu$), the sticking events appear to be Lévy flights in the reference frame co-moving with the mean position $\langle x \rangle$, and this accounts for the superdiffusive growth of the variance.

For the case when $1 < \alpha_1 < 2$, the first moment for the corresponding PDF is infinite. On average, all random walkers are undergoing their first flight (or sticking) event corresponding to this PDF. Thus, the growth in the variance comes from the rare random walkers that finish those events. If the second smallest PDF exponent α_2 is sufficiently small, γ depends on both α_1 and α_2 as shown in Fig. 2.

The results for the exponent γ are similar to those shown in Fig. 1(b), with the exception of the ballistic area for $\mu < 2$ in Fig. 1(b). The ballistic motion for the case with $\mu = \eta$ arises as the average random walker is undergoing its first flight even as $t \to \infty$, but the flight can either be leftward or rightward; for the strongly asymmetric case ($\mu \neq \eta$) one direction dominates. With $\mu = \eta$, random walkers going left diverge from those going right; with $\mu \neq \eta$, random walkers spend most of their time going in the same direction, and thus the divergence no longer occurs. Thus, for $\mu \neq \eta$ and either $1 < \mu < 2$ or $1 < \eta < 2$, the growth is no longer ballistic, and can in fact be subdiffusive (see Fig. 2).

The results can be easily extended to random walks with multiple types of flights, by slightly modifying the coefficients listed in the tables. All sums over the three types of events (leftward flights, rightward flights, sticking events) are modified to account for the additional flight types. Again, the variance exponent γ only depends on the PDFs with the slowest decay (smallest decay exponents α).

III. DISCUSSION

Exponentially decaying PDFs are common in physical situations [7]. All moments for an exponential PDF are finite, and the PDF can be treated as if the relevant power-law decay exponent were infinite. If both flight PDFs and the sticking PDF have exponential tails, the central limit theorem applies, and the behavior is normally diffusive. Cases without sticking events can be considered by taking the limits $\nu \to \infty$ and then $t_\nu \to 0$, in which $\langle t_{\text{stick}} \rangle = 0$, which slightly changes the coefficients given in Table II.

Our results have been derived for independent steps and pauses. The model could have been generalized to include effects of correlations, for example, correlations between successive steps, between the sticking duration and the direction of the subsequent flight, or between flight times and the subsequent sticking times [2]. However, in an analysis of our experimental data [5,7], we found no evidence of such correlations; hence we leave generalization of the model for future work.

Our results can be compared with the results obtained from the Hamiltonian model of Ref. [27]. This model describes a 2D flow consisting of a chain of vortices in a shear flow. Tracer particles alternate between being trapped in vortices and moving in jets. In several cases, flight times in the jets and sticking times in the vortices are well described by power-law PDFs. In particular, this Hamiltonian model yields different PDFs for leftward and rightward flights, and thus the results can be compared with our model.

Reference [27] considers three cases in detail, and finds for the exponents (μ, η, ν) the values $(\infty,\infty,2.9)$ [case (a)], $(3.26,\infty,2.4)$ [case (b)], and $(\infty,1.89,2.0)$ [case (c)] (where an exponent equal to ∞ represents non-power-law decay). The values of γ for cases (a), (b), and (c), respectively, are 1.42, 1.53, and 1.80, while the corresponding predictions of our analyses are 1.10, 1.60, and 1.89. The agreement is fair for cases (b) and (c) but not for (a). This suggests that perhaps the trajectories in the Hamiltonian model have hidden correlations, that is, that the motions in the jets and vortices are correlated. It is also possible that the asymptotic time limit has not been reached for case (a). For both cases (a) and (c), the exponent β in $\langle x \rangle \sim t^\beta$ is unity, in agreement with the predictions of our model. A value of β could not be determined for case (b).

In conclusion, we have investigated cases where the random walk alternates between sticking (motionless behavior), and flight (movement) behavior, where the flights consist of two distinct types. These results are qualitatively different from results [7] derived in situations with only one type of flight. In cases with Lévy flights where the first moment of the PDF is also infinite, an arbitrarily small difference in the decay of the flight PDFs changes the asymptotic behavior from ballistic ($\gamma = 2$) to superdiffusive ($\gamma > 1$) or even subdiffusive ($\gamma < 1$). This behavior can be understood by considering the first flight a random walker begins. If the first moment of the flight PDF is infinite, the average duration of this flight is infinite. The ratio of numbers of random walkers in flights with decay exponent η to walkers in flights with decay exponent μ is proportional to $t^{\mu-\eta}$ for large t; as $t \to \infty$ this ratio goes to zero if $\mu < \eta$ (or ∞ if $\mu > \eta$), and thus the behavior is dominated by the random walkers in the flight direction with the smaller decay exponent. In such a case the diffusion is anomalous, with $0 < \gamma < 2$. Only if $\mu = \eta$ does the behavior become ballistic with $\gamma = 2$, as the ratio of walkers in the two flight behaviors is a constant.

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